§1.E Examples of Hamiltonian Systems

(1E.1) Harmonic Oscillator.

$$M = P = T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n} \quad \text{phase space}$$

$$\omega = dq^j \wedge dp_j$$

$$H(q,p) = \tfrac{1}{2}\left(\|q\|^2 + \|p\|^2\right), \quad X_H = p_j \frac{\partial}{\partial q^j} - q^j \frac{\partial}{\partial p_j}$$

$$\dot{q} = p, \quad \dot{p} = -q$$

H is a first integral: Every motion $(q,p): I \to M$ with $H(q(t_0), p(t_0)) = E \geq 0$ satisfies

$$H(q(t), p(t)) = E \quad \text{for all } t \in I.$$

Hence, it remains in the hypersurface

$$\Sigma_E = H^{-1}(E).$$

Σ_E is a submanifold of dimension $2n-1$ since $\nabla H = (q,p) \neq 0$ for $E > 0$. We see

$$\Sigma_E = \left\{(q,p) \mid \|q\|^2 + \|p\|^2 = 2E\right\} = \mathbb{S}^{2n-1}(\sqrt{2E}),$$

where

$$\mathbb{S}^{k-1}(R) := \left\{x \in \mathbb{R}^k \mid \|x\|^2 = R^2\right\}$$

denotes the $(k-1)$-sphere of radius R.

(1E.2) Reduction with respect to first integrals

Let F be a first integral of a hamiltonian

system (M, ω, H), i.e. $F \in \mathcal{E}(M)$ and $\{F, H\} = 0$.
Let $c \in \mathbb{R}$ be a value of F with

$$\Sigma_c := F^{-1}(c) \neq \emptyset.$$

Assume that Σ_F is a smooth hypersurface, this holds e.g. if $\nabla F \neq 0$ on Σ_F. Then the space of orbits with $F=c$ is the quotient

$$O_c := \Sigma_c / \sim$$

with respect to the equivalence relation

$$a \sim b \iff \exists \text{ motion } x : I \longrightarrow \Sigma_c \text{ with } x(t_1) = a \ \& \ x(t_2) = b.$$

The orbit space O_c is a $(2n-2)$-dimensional manifold and $\omega|_{\Sigma_c}$ induces on O_c a natural symplectic form $\omega_c \in \Omega^2(O_c)$ (such that $\omega|_{O_c} = \pi^*(\omega_c)$ for the projection $\pi : \Sigma_c \to O_c$). Moreover, H descends to O_c as $H_c \in \mathcal{E}(O_c)$ with $H = H_c \circ \pi$ on Σ_c.

As a result, the original system (M, ω, H) has been reduced (by one degree of freedom) to (O_c, ω_c, H_c). In general, this procedure can be repeated. In good cases ("completely integrable systems"[2]) one can go down to 0-dimensional reduction which then gives the solution.

In case of the harmonic oscillator of dimension n, the orbit space $\Sigma_E/\sim$ is isomorphic to the complex projective space $\mathbb{P}^{n-1}(\mathbb{C})$ of all complex lines through $0 \in \mathbb{C}^n$ in $\mathbb{C}^n$.

Moreover, all the functions

$$H_j := \frac{1}{2}\left(p_j^2 + (q^j)^2\right), \quad j = 1, \dots n,$$

on $\mathbb{R}^{2n}$ are first integrals:

$$\frac{d}{dt} H_j(q(t), p(t)) = q^j \dot{q}^j + p_j \dot{p}_j \qquad \text{(no summation)}$$

$$= q^j p_j + p_j(-q^j) \qquad \text{"}$$

$$= 0$$

With $E = \Sigma E_j$, $E_j = H_j(q(t_0), p(t_0))$, $\vec{E} = (E_1, \dots E_n)$, we obtain

$$M_{\vec{E}} = \bigcap_{j=1}^{n} H_j^{-1}(E_j) = \prod_{j=1}^{n} S^1(\sqrt{2E_j})$$

and again, the motion $(q(t), p(t))$ remains in $M_{\vec{E}}$. This "reduction" gives a complete solution. Every motion $x = x(t) = (q(t), p(t))$, $x_j(t) = (q^j(t), p_j(t))$ satisfies $x_j(t) \in S^1(\sqrt{2E_j})$, is determined by $x(0)$ being of the form

$$x_j(t) = \left(q^j(0)\cos t + p_j(0)\sin t,\ p_j(0)\cos t - q^j(0)\sin t\right)$$

if $t_0 = 0$.

(1E.3) <u>Kepler problem</u> (hydrogen atom)

$$Q = \mathbb{R}^3 \setminus \{0\} \quad , \quad M = T^*Q = Q \times \mathbb{R}^4$$

$\omega = dq^j \wedge dp_j$ the usual 2-form

$$H(q,p) = \frac{1}{2m} \|p\|^2 - \frac{k}{\|q\|} \quad , m, k > 0.$$

We have $\nabla H = \left(\frac{kq}{\|q\|^3}, -\frac{p}{m} \right) \neq 0$ on all of M.
Hence, the energy hypersurface

$$\Sigma_E = H^{-1}(E)$$

is a smooth submanifold of dimension 5 for all $E \in \mathbb{R}$.

Let $E \in]-\infty, 0[$. The orbits in Σ_E are ellipses and one can show, that the orbit space $O_E = \Sigma_E/\sim$ is isomorphic to $S^2(mk) \times S^2(mk)$. The symplectic form ω descends to ω_E on $\Sigma_E/\sim$. And on $S_E := S^2(mk) \times S^2(mk)$ it has the form

$$\frac{1}{2\varrho} \left(\frac{dx_1 \wedge dx_2}{x_3} + \frac{dy_1 \wedge dy_2}{y_3} \right),$$

with respect to the chart with $x_3 \neq 0 \neq y_3$ on S_E. Here $\varrho = \sqrt{-2mE}$.

Which energy values occur if we quantize the system (S_E, ω_E) according to the program of geometric quantization?

Result: $E_N = -2\pi^2 m k^2 N^{-2}$, $N \in \mathbb{N}, N \geq 1$, the values known from experiments!

(1E.4) Relativistic charged particle

Q spacetime with (Lorentzian metric) e.g. Minkowski space $\mathbb{R}^4$

$M = T^*Q$

F 2-form on Q of electromagnetic field (locally $F|_U = dA$, for $A \in \Omega^1(U)$)

e charge of particle

$\omega_e := \omega_0 + e\,\tau^*(F)$, $\omega_0 = dq^j \wedge dp_j$ on M.

(1E.5) <u>Particle with spin</u>

(1E.6) <u>Kähler manifolds</u>

Symplectic manifolds with additional complex structure are the Kähler manifolds which we describe later in detail. Prominent examples are the projective spaces $\mathbb{P}^n(\mathbb{C})$ and their closed complex submanifolds: the compact algebraic Kähler manifolds.

(1E.7) Coadjoint Orbits.

This class of examples of symplectic manifolds yields a close connection the representation theory of Lie groups and Lie algebras with geometric quantization (close but not obvious).

In the following:

G is a connected Lie group (for example a closed matrix group $G \subset GL(k, \mathbb{R})$) of finite dimension.

$\mathfrak{g} := \mathrm{Lie}\, G$ is the associated Lie algebra,

the conjugation with $g \in G$ is the smooth map

$$\tau_g : G \longrightarrow G \;,\; x \longmapsto g x g^{-1},\; x \in G,$$

and the ADJOINT REPRESENTATION is defined as

$$Ad_g := T_e \tau_g : T_e G = \mathfrak{g} \longrightarrow \mathfrak{g} = T_e G .$$

In fact, $A_{gh} = Ad_g \circ Ad_h$ for $g, h \in G$, so that

$$Ad : G \longrightarrow GL(\mathfrak{g})$$

is a Lie group homomorphism.

Definition: The COADJOINT REPRESENTATION is the "dual" or "adjoint" of the adjoint representation: $Ad^* : G \longrightarrow GL(\mathfrak{g}^*)$ is given by

$$\mathrm{Ad}_g^*: \mathfrak{g}^* \longrightarrow \mathfrak{g}^*, \; \mu \longmapsto \mathrm{Ad}_g^*(\mu) \in \mathfrak{g}^*,$$

with
$$\mathrm{Ad}_g^*(\mu)(X) := \mu\left(\mathrm{Ad}_{g^{-1}}(X)\right)$$
for $\mu \in \mathfrak{g}^* = \{ \nu : \mathfrak{g}^* \to \mathbb{R} \mid \mathbb{R}\text{-linear} \}$
and $X \in \mathfrak{g}$.

It is easy to check that $A^*_{gh} = \mathrm{Ad}_g^* \circ \mathrm{Ad}_h^*$, $g,h \in G$,
i.e. A^* is a Lie group homomorphism.

As a result we have an action of G on $\mathfrak{g}^*$

$$G \times \mathfrak{g}^* \to \mathfrak{g}^* \quad , (g,\mu) \longmapsto \mathrm{Ad}_g^*(\mu)$$

(the coadjoint action) with orbits
$$M_\mu := \{ \mathrm{Ad}_g^* \mu \mid g \in G \} \subset \mathfrak{g}^*$$

One can show:

1) M_μ is a smooth submanifold of $\mathfrak{g}^* \cong \mathbb{R}^n$ with a natural symplectic form ω_μ and with symmetry group G.

2) Every symplectic manifold M on which G acts transitively by symplectomorphisms looks locally like a suitable orbit M_μ such that $M \to M_\mu$ is a covering.

Here, $\varphi : (M,\omega) \to (M',\omega')$ is a symplectomorphism if φ is a diffeomorphism with $\varphi^* \omega' = \omega$.

(1E.8) Lagrange mechanics

Q n-dimensional manifold,

$M = TQ$ velocity phase space,

$L \in \mathcal{E}(M)$ Lagrange function.

Definition: $q : I \to Q$ is a MOTION of the Lagrange system (M, L), if q satisfies the Euler Lagrange equations. That is

$$\dot{q} = v \in TQ = M \quad \text{and}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = \frac{\partial L}{\partial q} \quad \text{(in bundle coordinates).}$$

Given a chart

$$\varphi = (q^1, \ldots q^n) : U \to V \subset \mathbb{R}^n$$

with associated bundle chart (cf. §1)

$$\tilde{\varphi} = (q^1, \ldots q^n, v^1, \ldots v^n) : TU \longrightarrow V \times \mathbb{R}^n$$

the local 1-form induced by L is

$$\lambda_L := \frac{\partial L}{\partial v^k}\, dq^k \quad \left(\frac{\partial L}{\partial v^k} \text{ "gen. moments"}\right)$$

defining a 2-form

$$\omega_L := -d\lambda_L$$
$$\omega_L = -\frac{\partial^2 L}{\partial q^j \partial v^k}\, dq^j \wedge dq^k - \frac{\partial^2 L}{\partial v^j \partial v^k}\, dv^j \wedge dq^k$$

ω_L is well-defined on all of M and it is closed. Therefore:

ω_L is a symplectic form

$\Leftrightarrow$ ω_L non-degenerate

$\Leftrightarrow$ L regular

$$:\Leftrightarrow \det\left(\frac{\partial^2 L}{\partial v^j \partial v^k}\right) \neq 0 .$$

Hence, for regular L (M, ω_L) is a symplectic manifold and (M, ω_L, H_L) is a hamiltonian system with the same motions as (M, L).

Legendre transformation:

$$L_q := L|_{T_qQ} : T_qQ \to \mathbb{R} \quad , q \in Q, \text{ with}$$

$$DL_q : T_qQ \to (T_qQ)^* = T_q^*Q \quad (DL_q \text{ derivative})$$

$$\rho_L(X) := DL_q(X) \qquad \text{"fibre derivative"}$$

Locally: $\rho(X)|_U = \frac{\partial L}{\partial v^j}(X)\, dq^j$

Pullback:

- $\rho_L^*(q^j) = q^j \quad , \rho_L^*(p_j) = \frac{\partial L}{\partial v^j}$
- $\rho_L^*(\lambda) = \lambda_L$
- $\rho_L^*(\omega) = \omega_L$

and: L regular $\Leftrightarrow$ ρ locally diffeomorph.

$H_L \rightsquigarrow X_{H_L}$ uniquely in that case.